

# Tunneling-induced damping of phase coherence revivals in deep optical lattices

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We consider phase coherence collapse and revival in deep optical lattices, and calculate within the Bose-Hubbard model the revival amplitude damping incurred by a finite tunneling coupling of the lattice wells (after sweeping from the superfluid to the Mott phase). Deriving scaling laws for the corresponding decay of first-order coherence revival in terms of filling factor, final lattice depth, and number of tunneling coupling partners, we estimate whether revival-damping related to tunneling between sites can be or even has already been observed in experiment.

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*Introduction.* The idea of implementing the Bose-Hubbard model with cold atoms in optical lattices ultimately led to a clean experimental realization of the superfluid-Mott transition [1, 2]. This in turn inspired many intriguing developments marrying condensed matter physics, quantum optics, and quantum information processing [3]. Concurrent with the superfluid-Mott transition, revivals of first-order phase coherence upon entering the Mott phase in a 3D optical lattice have been observed [4]. In the ideal case of a perfectly homogeneous lattice, the revivals would be exact, i.e., reach the same coherence amplitude as the initial superfluid state, provided the optical lattice wells were completely isolated from each other and from the environment (heat bath). The experiment [4] however measures a significant damping of the revival amplitude (so that approximately four to five revivals can be observed clearly). Here, the revival time  $t_{\text{rev}}$  is defined to be the time at which the first revival maximum of the long-range coherence  $\langle \hat{a}_\mu^\dagger(t) \hat{a}_\sigma(t) \rangle$  occurs after the sweep to the Mott phase. There are basically four mechanisms which might induce the observed damping: *i*) a finite tunneling coupling  $J$  of the lattice wells, *ii*) a finite coupling of the wells to the environment, and finally inhomogeneities in the lattice, such as *iii*) an external trap potential (inducing finite-size effects), and *iv*) variations of the lattice depth and hence on-site repulsion  $U$  from site to site (i.e., diagonal disorder). Here, we investigate the question of whether and how a finite tunneling coupling  $J$  of lattice wells induces a damping of the revival signal, i.e., mechanism *i*). In contrast to the other effects, this mechanism is an intrinsic property of the system and survives in the homogeneous continuum limit. It is also theoretically well under control and of fundamental interest, e.g., regarding the basic problem of equilibration, that is the question of whether and how fast the system approaches a local equilibrium state after the quench to the Mott phase. We point out, as these arguments already indicate, that the nature of revivals and their damping in an optical lattice is fundamentally different from the single-well case where the particle-number dependence of the chemical potential plays the major role [5]. Our strategy is to calculate perturbatively the solution of the many-body problem on a

general lattice, when the parameter  $2\pi J/U$  is small but finite, starting from the exact number basis solution at  $J = 0$ . Here  $J$  gives a typical scale of the hopping amplitude, and  $U$  is proportional to the contact interaction coupling. We note that within the presently employed perturbative approach, we cannot discuss the late-time behavior but can extract the decay of first-order coherence from one given to the next revival cycle.

*Tunneling-induced damping of revivals.* The Hamiltonian we employ is of the conventional single-band Bose-Hubbard type with on-site interactions

$$\hat{H} = JM_{\bar{\mu}\bar{\nu}}\hat{a}_{\bar{\mu}}^\dagger\hat{a}_{\bar{\nu}} + \frac{U}{2}(\hat{a}_{\bar{\mu}}^\dagger)^2\hat{a}_{\bar{\mu}}^2, \quad (1)$$

where  $\hat{a}_\mu^\dagger$  and  $\hat{a}_\nu$  are bosonic creation and annihilation operators at lattice sites  $\mu$  and  $\nu$ , respectively. We use the convention that summation is implicit over the equal Greek lattice site indices designated with an overbar. The matrix  $M_{\mu\nu}$  describes the (possibly anisotropic) tunneling rates on an arbitrary lattice [6], and  $U$  is the on-site repulsion. The full operator equation of motion in the final state with given  $J$  then reads ( $\hbar = 1$ )

$$i\frac{d\hat{a}_\mu}{dt} = JM_{\mu\bar{\nu}}\hat{a}_{\bar{\nu}} + U\hat{n}_\mu\hat{a}_\mu, \quad (2)$$

where  $\hat{n}_\mu = \hat{a}_\mu^\dagger\hat{a}_\mu$  counts the number of particles per site. The exact solution of the above operator equation, for decoupled wells,  $J = 0$ , is given by

$$\hat{a}_\mu^0(t) = \exp\{-iU\hat{n}_\mu t\}\hat{a}_\mu^0(0). \quad (3)$$

The revival time, where the argument of the exponential operator assumes integer multiples of  $2\pi$ , is  $t_{\text{rev}} = 2\pi/U$  (for  $J = 0$ ). These coherence revivals exist because of the discrete spectrum of the filling operator  $\hat{n}_\mu$ , i.e., because of the existence of particles, the “granularity” of matter. Calculations for collapse and revival of first- and higher-order correlation functions for the decoupled-well case  $J = 0$  were performed in [8]. We mention here that in the opposite limit of hard-core bosons ( $U \rightarrow \infty$ ), hopping in an optical lattice from site to site, phase coherence collapse and revival may occur when an optical superlattice is switched on [9].

We now proceed to calculate the perturbative solution of (2) for  $J/U$  small but finite. Defining

$$\hat{a}_\mu(t) = \exp\{-iU\hat{n}_\mu t\}\hat{A}_\mu(t), \quad (4)$$

we extract the dominant (for  $J/U \ll 1$ ) time-dependence from the full field operator  $\hat{a}_\mu(t)$ , so that  $\hat{A}_\mu(0) = \hat{a}_\mu(0)$  and  $\hat{A}_\mu(t_{\text{rev}}) = \hat{a}_\mu(t_{\text{rev}})$ . The equation of motion

$$i\partial_t \hat{A}_\mu = JM_{\mu\bar{\nu}} \exp\{iU(\hat{n}_\mu - \hat{n}_{\bar{\nu}})t\}\hat{A}_{\bar{\nu}} \quad (5)$$

can be solved via an expansion into powers of the small parameter  $J$  (analogous to response theory with a perturbation  $\hat{H}_{\text{int}} = JM_{\mu\bar{\nu}}\hat{a}_\mu^\dagger\hat{a}_{\bar{\nu}}$ ), where the second-order solution reads

$$\begin{aligned} \hat{A}_\mu(t) = & \hat{A}_\mu(0) - i \int_0^t dt' JM_{\mu\bar{\nu}} e^{iU(\hat{n}_\mu - \hat{n}_{\bar{\nu}})t'} \hat{A}_{\bar{\nu}}(0) \\ & - \int_0^t dt' \int_0^{t'} dt'' J^2 M_{\mu\bar{\nu}} M_{\bar{\nu}\bar{\rho}} e^{iU(\hat{n}_\mu - \hat{n}_{\bar{\nu}})t'} e^{iU(\hat{n}_{\bar{\nu}} - \hat{n}_{\bar{\rho}})t''} \hat{A}_{\bar{\rho}}(0) \\ & + \mathcal{O}(J^3). \end{aligned} \quad (6)$$

Inserting the above second-order solution into the first-order correlation function evaluated at time  $t_{\text{rev}}$  yields

$$\begin{aligned} \langle \hat{A}_\mu^\dagger(t_{\text{rev}}) \hat{A}_\sigma(t_{\text{rev}}) \rangle = & \langle \hat{A}_\mu^\dagger(0) \hat{A}_\sigma(0) \rangle - \left( \frac{2\pi J}{U} \right)^2 \times \\ & \times \left\{ \langle \hat{A}_{\bar{\nu}}^\dagger(0) \delta(\hat{n}_\mu - \hat{n}_{\bar{\nu}}) \delta(\hat{n}_\sigma - \hat{n}_{\bar{\rho}}) \hat{A}_{\bar{\rho}}(0) \rangle M_{\mu\bar{\nu}} M_{\bar{\rho}\sigma} \right. \\ & + \frac{1}{2} \langle \hat{A}_\mu^\dagger(0) \delta(\hat{n}_{\bar{\nu}} - \hat{n}_{\bar{\rho}}) \delta(\hat{n}_\sigma - \hat{n}_{\bar{\rho}}) \hat{A}_{\bar{\nu}}(0) \rangle M_{\bar{\nu}\bar{\rho}} M_{\bar{\rho}\sigma} \\ & \left. + \frac{1}{2} \langle \hat{A}_{\bar{\rho}}^\dagger(0) \delta(\hat{n}_\mu - \hat{n}_{\bar{\nu}}) \delta(\hat{n}_{\bar{\nu}} - \hat{n}_{\bar{\rho}}) \hat{A}_\sigma(0) \rangle M_{\mu\bar{\nu}} M_{\bar{\nu}\bar{\rho}} \right\} \\ & + \mathcal{O}(J^3), \end{aligned} \quad (7)$$

where  $\delta(\hat{n}_\nu - \hat{n}_\rho)$  is to be understood as an operator Kronecker delta, and the average is taken with respect to the initial many-body quantum state. The above expression represents the full quantum result to second order in  $J$ , which is general insofar, within the single-band Bose-Hubbard model. This should be contrasted with the approach of [11], which considers quantum collapse and revival in a two-dimensional (2D) respectively three-dimensional (3D) optical lattice, using resonant tunneling in the reduced dynamics of a two-mode (2D) or three-mode (3D) model.

The terms linear in  $J$  generally vanish in the final result for the correlator (7), which is to be expected because  $\langle \hat{A}_\mu^\dagger(0) \hat{A}_\sigma(0) \rangle$  assumes its maximum value ( $\equiv n$  in the homogeneous case) in a coherent state. Within linear response, the tunneling-induced damping therefore generally vanishes. We now use such a coherent state, more precisely a product of coherent states at each site, which has Poissonian number statistics, to evaluate (7),

$$|\text{coh}\rangle = \prod_\mu |\alpha\rangle_\mu = \prod_\mu e^{-|\alpha_\mu|^2/2} \sum_{n_\mu=0}^{\infty} \frac{\alpha_\mu^{n_\mu}}{\sqrt{n_\mu!}} |n_\mu\rangle \quad (8)$$

where  $|n_\mu\rangle$  are local number eigenstates,  $\hat{n}_\mu|n_\mu\rangle = |n_\mu\rangle n_\mu$  and  $\hat{a}_\mu|\alpha\rangle_\mu = |\alpha\rangle_\mu \alpha_\mu$  with Poissonian distribution and average  $\langle \hat{n}_\mu \rangle = |\alpha_\mu|^2$ . The state (8) is the factorized many-body state after a quasi-instantaneous sweep starting deep in the superfluid state [6]. We assume homogeneity (no external trapping),  $\alpha_\mu = \alpha$ ,  $\langle \hat{n}_\mu \rangle = n$ , and calculate the two different terms occurring in the correlator (7). For the relevant case of long-range coherence (i.e., the two lattice sites  $\mu$  and  $\sigma$  are far apart and do not share neighbors, the result then being independent of the distance of the two sites), the first term gives

$$\begin{aligned} \langle \hat{A}_{\bar{\nu}}^\dagger(0) \delta(\hat{n}_\mu - \hat{n}_{\bar{\nu}}) \delta(\hat{n}_\sigma - \hat{n}_{\bar{\rho}}) \hat{A}_{\bar{\rho}}(0) \rangle M_{\mu\bar{\nu}} M_{\bar{\rho}\sigma} = \\ = n \left( \sum_{k=0}^{\infty} p_{k,n}^2 \right)^2 D^2 = n e^{-4n} I_0^2(2n) D^2, \end{aligned} \quad (9)$$

where  $D$  is the number of neighboring sites defined by  $D = \sum_\nu M_{\mu\nu}$  for a fixed site  $\mu$ , with the  $\nu$ -sum running in the simplest case over only nearest neighbors to site  $\mu$  (see however, discussion below of including next-nearest neighbors as well), and  $I_0$  is modified Bessel function; the Poisson probabilities to find  $k$  particles in a coherent state with the number average  $n$  are denoted  $p_{k,n} = e^{-n} n^k / k!$ . Similarly (for long-range coherence), the last two terms in (7) each give a contribution

$$\begin{aligned} \frac{1}{2} \langle \hat{A}_\mu^\dagger(0) \delta(\hat{n}_{\bar{\nu}} - \hat{n}_{\bar{\rho}}) \delta(\hat{n}_\sigma - \hat{n}_{\bar{\rho}}) \hat{A}_{\bar{\nu}}(0) \rangle M_{\bar{\nu}\bar{\rho}} M_{\bar{\rho}\sigma} \\ = \frac{1}{2} n \left( \sum_{k=0}^{\infty} p_{k,n}^2 \right)^2 D(D-1) + \frac{1}{2} n \sum_{k=0}^{\infty} p_{k,n}^3 D \\ = \frac{n}{2} e^{-4n} I_0^2(2n) D(D-1) + \frac{n}{2} e^{-3n} F_{\{1,1\}}(n^3) D, \end{aligned} \quad (10)$$

where  $F_{\{1,1\}}(n^3)$  is a hypergeometric function. The second term is generally negligible for  $D \gg 1$ , cf. Fig. 1, because  $\sum_{k=0}^{\infty} p_{k,n}^3 \sim (\sum_{k=0}^{\infty} p_{k,n}^2)^2$  for the experimentally relevant range of  $n = 1 \dots 10$ , with the ratio of  $\sum_{k=0}^{\infty} p_{k,n}^3$  to  $(\sum_{k=0}^{\infty} p_{k,n}^2)^2$  slowly decreasing with increasing  $n$ .

Collecting the correlation functions from (9) and (10), the total result for the decay of first order coherence at the instant of the first revival reads (see also Fig. 1)

$$\begin{aligned} \Delta(t_{\text{rev}}) \equiv \frac{\langle \hat{A}_\mu^\dagger(0) \hat{A}_\sigma(0) \rangle - \langle \hat{A}_\mu^\dagger(t_{\text{rev}}) \hat{A}_\sigma(t_{\text{rev}}) \rangle}{n} = \\ = \left( \frac{2\pi J}{U} \right)^2 [D(2D-1) e^{-4n} I_0^2(2n) + D e^{-3n} F_{\{1,1\}}(n^3)]. \end{aligned} \quad (11)$$

This represents our major result for a coherent state with Poissonian statistics, i.e., after a sudden sweep from deep within the superfluid phase, for which the original ground state (8) has no time to adjust.

*Numerical estimates and discussion.* We now provide a numerical estimate of (11) for the experiment

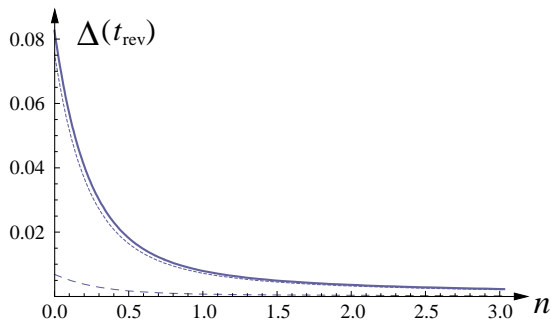


FIG. 1: The result (11) for the damping rate as a function of the filling  $n$  (bold line), for  $(2\pi J/U)^2 = 1.15 \times 10^{-3}$  and  $D = 6$  [4]. Dotted and dashed lines represent the contributions of the  $I_0^2$  and  $F_{\{1,1\}}$  terms in (11), respectively.

[4], which was performed in a cubic 3D optical lattice, with an initial depth  $V_A = 8E_r$  (corresponding to the superfluid phase), and a subsequent “jump” to a final depth  $V_B = 22E_r$  (deep into Mott phase). In terms of the recoil energy  $E_r = 2\pi^2/(m\lambda^2)$  determined by the laser wavelength  $\lambda$ , the tunneling coupling  $J$  in sufficiently deep optical lattices can be estimated by  $J/E_r = 8(V/E_r)^{3/4} \exp[-2\sqrt{V/E_r}]/\sqrt{\pi}$ . Similarly, the on-site repulsion scales as  $U/E_r = 4\sqrt{2\pi}(V/E_r)^{3/4}a_s/\lambda$ , where  $a_s$  is the  $s$ -wave scattering length [7]. One obtains for the perturbative parameter of our calculation

$$\left(\frac{2\pi J}{U}\right)^2 = \frac{8\lambda^2}{a_s^2} \exp\left[-4\sqrt{\frac{V}{E_r}}\right]. \quad (12)$$

Assuming, like for  $^{87}\text{Rb}$ ,  $a_s = 6$  nm, and a laser wavelength of  $\lambda = 852$  nm, we have for  $V = V_B = 22E_r$  the value  $(2\pi J/U)^2 = 1.15 \times 10^{-3}$ . With the number  $D = 6$  of nearest neighbors in a 3D cubic lattice and the experimental value [4] of the average filling,  $n = 2.5$ , we obtain  $\Delta(t_{\text{rev}}) = 2.8 \times 10^{-3}$ . Thus damping induced by tunneling to nearest neighbors, evaluated on the basis of a superfluid state with Poissonian statistics cannot directly explain the amount of damping observed in experiment [4], which is one to two orders of magnitude larger [10].

The result (11) however represents a lower bound for tunneling-induced damping. We now discuss various scenarios which would increase  $\Delta(t_{\text{rev}})$  above this lower bound. Firstly, (11) is the result obtained using the Poissonian superfluid state ( $J \gg U$ ) as the initial unperturbed (zeroth-order) state. Assuming sub-Poissonian number statistics with a reduced number variance increases the expectation values of Kronecker symbols  $\delta(\hat{n}_\mu - \hat{n}_\nu)$ . In view of  $\langle \psi | \delta(\hat{n}_\mu - \hat{n}_\nu) | \psi \rangle \leq 1$  for all states  $|\psi\rangle$ , we may obtain a crude upper estimate by just omitting all Kronecker symbols  $\delta(\hat{n}_\mu - \hat{n}_\nu)$  in the expectation values. Inserting this upper bound, reasonable agreement with experiment can be reached, noting that  $\sum_{k=0}^{\infty} p_{k,2.5}^3 \simeq 0.034$  and  $(\sum_{k=0}^{\infty} p_{k,2.5}^2)^2 \simeq 0.0385$ , which would then essentially be replaced by unity. Note, however, that actually realizing this upper bound would

imply that the many-particle state approaches a product of Fock states with exactly  $n$  particles at each site – for which the coherence signal  $\langle \hat{A}_\mu^\dagger \hat{A}_\sigma \rangle$  vanishes. Nevertheless, a state “in-between” the coherent state and the Mott state could still display non-vanishing long-range coherence (though below its maximum),  $n > \langle \hat{A}_\mu^\dagger \hat{A}_\sigma \rangle > 0$ , whose revivals are more strongly damped than in the fully coherent case (11). Actually, such an intermediate state with sub-Poissonian number statistics is automatically created by a superfluid-Mott quench with a *finite* sweep rate [6]. The character of the final state (i.e., whether it is closer to the initial superfluid phase or the Mott state) depends on the time scale of the quench in comparison with the (inverse) chemical potential, i.e., the relevant internal energy scale [6]. From the revival time of  $550 \mu\text{sec}$  [4], one obtains for  $n = 2.5$  an inverse chemical potential of  $\mu^{-1} = (Un)^{-1} = 40 \mu\text{sec}$ . According to [4], the system is quenched in  $50 \mu\text{sec}$  from superfluid to Mott. The initial  $V_0/E_r = 8$  and final values  $V_0/E_r = 22$  imply that  $J$  is quenched across approximately 3.7  $e$ -foldings. In order to compare the results with the calculation in [6], we assume an exponential decay  $J(t) \propto \exp\{-\gamma t\}$  for simplicity. This implies that  $1/\gamma \simeq 13.5 \mu\text{sec}$ , which is about a factor of three smaller than the inverse chemical potential of  $\mu^{-1} = (Un)^{-1} = 40 \mu\text{sec}$ . Therefore, the sweep is rather non-adiabatic, which corresponds to a small adiabaticity parameter  $\nu = Un/\gamma \approx 1/3$  introduced in [6]. However, the sweep is not sudden – for a sudden quench, the number fluctuations would essentially retain their initial value,  $\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = \langle \delta \hat{n}^2 \rangle = n$  for a coherent state. Due to the finite sweep rate  $J(t) \propto \exp\{-\gamma t\}$  with  $\nu = Un/\gamma \approx 1/3$ , though, the number variance is reduced by 60%, i.e.,  $\langle \delta \hat{n}^2 \rangle \approx 0.4n$ . Even though this result, derived in [6] applies, strictly speaking, in the limit of large fillings  $n \gg 1$  only, one would expect a similar sub-Poissonian statistics also for  $n = 2.5$ , which would then increase the decay  $\Delta(t_{\text{rev}})$  in (11) significantly. An additional source for sub-Poissonian statistics could be the initial state itself, if we do not start deep in the superfluid phase. Closer to the transition line (but still on the superfluid side), the number variance decreases,  $\langle \delta \hat{n}^2 \rangle < n$  already in the initial ground state. (In the experiment [4], merely 60% of the atoms occupied the coherent condensate state initially.)

Another effect which potentially increases  $\Delta(t_{\text{rev}})$  is to include couplings to next nearest neighbors, i.e., to effectively increase  $D$ . For the parameters of the experiment [4], the final next-nearest-neighbor couplings may be estimated as follows: Within a harmonic approximation, the tunneling matrix elements are given by  $JM_{\alpha\beta} \simeq J_{|\alpha-\beta|} \exp\{-\pi^2 \sqrt{V_0/E_r}(\alpha-\beta)^2/4\}$ , where  $|\alpha-\beta|$  is the distance of lattice points  $\alpha$  and  $\beta$  in units of the lattice spacing and  $J_{|\alpha-\beta|}$  depends polynomially on  $|\alpha-\beta|$  [12, 13]. There are 20 next-nearest (diagonal) neighbors in a 3D cubic lattice in addition to the 6 nearest (straight) neighbors. Due to the exponential reduction of their individual contributions when quenching deep into Mott phase ( $V_0/E_r = 22$ ), their contribution gives about

a factor of two for the damping  $\Delta(t_{\text{rev}})$ . On the other hand, using only a moderately smaller final  $V_0/E_r$  may make their total contribution to revival damping larger than that of the nearest neighbors.

*Concluding remarks.* In conclusion, we have derived a rigorous second-order result (7) for tunneling-induced damping of phase coherence revivals in optical lattices sufficiently deep inside the localized Mott phase  $2\pi J \ll U$ . Evaluating expression (7) for a coherent state, we obtained a lower bound for tunneling-induced damping (11), which is too small to explain the experiment [4]. However, the incorporation of next-nearest (diagonal) neighbors as well as sub-Poissonian number statistics induced by the initial state and the finite sweep rate significantly enhance the tunneling-induced damping  $\Delta(t_{\text{rev}})$  such that – even though it does perhaps not fully reproduce the experiment [4] – it should constitute an observable fraction of the measured decay. Unfortunately, the precise value of the tunneling-induced damping of phase coherence cannot be derived from the information given in [4] since part of the relevant input – such as the exact dynamics  $J(t)$  – is missing. Thus, while the dephasing mechanisms *i*) to *iv*) discussed in the Introduction are qualitatively well understood, it is not possible to precisely disentangle their quantitative contributions from the data at hand.

While our prediction cannot be compared accurately to existing experimental results, a relatively modest modification of the parameters in an experiment like that of [4] will allow for its test. In particular, the exponential de-

pendence of the tunneling-induced damping  $\Delta(t_{\text{rev}}) \propto J^2$  on the laser amplitude  $\sqrt{V_0/E_r}$ , implying 7.4 *e*-foldings for  $(2\pi J/U)^2$  in the experiment [4], distinguishes (at low temperatures) this mechanism *i*) from other sources like inhomogeneities due to external trapping *iii*).

We finally point out the importance of tunneling coupling for the equilibration of many-body states on the lattice. Nonequilibrium states of the Bose-Hubbard model were studied for various cases, e.g., quenching from the superfluid to the Mott side [6, 15]; from Mott to superfluid [16]; or for hard-core bosons in superlattices [9]. While our approach is perturbative, yielding first-order coherence up to second order in  $2\pi J/U$ , it provides a first estimate on how important coupling of the wells can turn out to be for an equilibrium or non-equilibrium state to be established after a quench. Taking the decay of the first-order coherence as an indicator for locally approaching the equilibrium state, we conclude that equilibration occurs faster for many neighbors and for a larger tunneling rate (as one would expect) but also for sub-Poissonian number variance – this is somewhat surprising, as one would generally expect that a state which is already closer to the Mott state decays slower.

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